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FRESHMAN ALGEBRA AND THE AVERAGE FRESHMAN.

By Ella Durgin Gray.

Let us first consider our pupils in two classes, those for whom algebra has a practical value and those for whom it is merely a culture study. Now, as probably most of us agree, the practical value of algebra, in the sense of the man in the street, is almost negligible. Personally, apart from my professional connection with it, I can recall but one instance of its having been of service to me. If by practical value we mean its value to those who practise it, our first class is limited to those pupils who will use algebra in other courses, at high school or at college. Those of this class who elect review algebra in high school, usually find that their grade in the review has been largely determined during their freshman year by the degree of facility they acquired in interpreting and manipulating notations and by the clearness with which they distinguished certain principles and their applications. For the time allowed review algebra is generally adequate only for those who have already some insight and skill to test these in more complicated and lengthy exercises and to extend somewhat the ground previously covered. The advanced pupil who is hazy and uncertain about fundamental notions and procedures is himself at a great disadvantage and he retards greatly the progress of the others. The advanced pupil who has acquired a merely mechanical proficiency frequently has too large an estimate of his knowledge and therefore does not present so open a mind towards efforts to clarify, deepen and broaden his vision as he would in the discussion of new matter. To the college preparatory student, then, his freshman algebra is highly important.

While by far the majority of our pupils are not among those for whom algebra has a practical value, yet with present opportunities in evening, part-time, university extension, and correspondence courses, in mathematics, pure and applied, one never can be sure that any pupil may not later have practical need of algebra in the sense that has been defined. Every pupil, then, has a potential claim to that same grasp of the subject matter that we owe the college preparatory student.

However, for those not professedly in the first class, algebra is made a required study on account of the mental training it affords. What is the nature of this mental training? An obdurate and tiny boy once, in an after session conference, reiterated, to all my efforts to enlist his interest in his work, " I ain't got no use for algebra," and I was moved to ask myself, "Has he? What is the use of algebra to those in the second class? Would I, for instance, want some little girl for whom I cared to give her time to it daily unless it was to yield her other than such returns as powers of accuracy and persistence which she could gain from other subjects, richer to her in content; particularly, if the study was to mean to her daily toil, with the contracted muscles that are the reflex from a perturbed spirit?" And considering these questions, I have felt convinced that algebra affords a pleasurable medium wherein the average pupil may acquire, in greater or less degree, certain of those mental powers that in exceptional pupils have already developed, naturally or through training, and that these powers, reacting directly upon the pupil's comprehension of the subject matter, make possible greater practical value to him who needs it. Each exercise in algebra—each example, so called—constitutes for the pupil a unit of achievement and here, as in many games, the satisfaction of attainment is both an appeal to interest and a motive force. In making it possible for a pupil to score well in these units, we transform him from a listless taskman to one who exercises his activities the more because of satisfaction through doing so.

By reviewing some of these respects in which algebra affords opportunity for mental development, we may, perhaps, agree that for the advantage of each class of pupils, those for whom algebra has practical value and those for whom it is merely cultural, the same subject matter, the same class work, gives each his due. If the development of mental power among those in the first class enables them to make the content of greater practical value, and if the content of algebra furnishes the freshmen of the second class with the handiest substance available to them for sharpening their infant mental incisors, then the distinction made between these two classes results in our clarifying our own understanding of what is for the good of each.

What, then, are the grounds in which algebra is fertile for mental growth? One, in which it sometimes does not yield all it might, is that for cultivating the power of discrimination between the accidental and the characteristic. How frequently our pupils appear to understand a principle, applying it with some accuracy when first it is presented to them, and while the exercise are all of a kind, only to become much entangled later when it occurs in a miscellaneous set, or in a complicated or a literal expression. They do not see beneath the form, nor always all of the form. A marked instance of this failure to discriminate occurs when the class reaches the chapter on factoring. The average pupil recites glibly as case after case is taken up, but when confronted with a review set, he confuses the various types, lamentably. If $9a^2c - c^5$ occurs among some examples of the binomial that is the difference of two squares, he is likely to give as the factors of $9a^2c - c^5$, some binomial difference and some binomial sum. Whether $9a^2c - c^5$ is itself a difference of two squares and, if so, what are the roots of these squares, are not matters of interest or inquiry. The number of terms and the arangements of signs, both in the product and in the factors, appear to have been for him the salient features in the type of the difference of two squares. It is possible here with gain to him in the double sense before noted, that of hold on subject matter and of mental power, to set before him, at the time of his first acquaintance with the chapter on factoring, all the main types simultaneously and to force him from the start to discriminate among them. For about five years I have tried this with very satisfactory concrete results and in the belief that this method is more educational. It needs to be reinforced by presenting these types both in formulas and in precisely worded descriptions and with both simple and fairly difficult illustrations and by training the pupils to prove their factoring by mental multiplication and thus find their errors as they make them, and it needs to be preceded by a preparatory exercise in which is another opportunity for discrimination. Many pupils have only vague notions as to what are squares and what are cubes. After they have formed tables of the squares and cubes of roots, say from $2a^2$ to $10a^{10}$, they can profitably drill in naming as squares, as cubes, as both, as neither, such expressions as 9, a^9, $9a^2$, $9a^2c$, a^6, 6, 1, $8a^9$, $9a^8$, $64a^6$, $64a^9$, etc. Even if the chapter on factoring was not an essential one, it would be valuable for the practice it may afford in observing and classifying characteristics.

Algebra furnishes an extensive field for exercise of judgment. A pupil working on simultaneous linear equations can readily be persuaded of the advantage of keeping his numerical coefficients as small as possible, and can become accustomed to decide before entering upon elimination, which elimination will involve the smaller multipliers. In literal simultaneous equations of the first degree, he has a chance for decision as to whether it will involve less labor to obtain the second unknown by substitution of the first or to get it as he got the first. In simplifying a complex fraction, it is frequently worth his while to make a choice between treating it as a quotient of polynomials, such as he may already have simplified in his study of division of fractions, and regarding it merely as a fraction, to be simplified by transforming both terms by means of a suitable multiplier. Again, in reducing certain complicated expressions in the chapter on the theory of exponents, he can be shown that it costs him time and effort not to consider whether he had better simplify through regarding each as a case of $(a^m)^n$, of $a^{m/n}$ or a^{-n}, etc. So many expressions in this chapter may be simplified by several methods. $a^{-1/2}b^{3/4}c$ times $a^{1/2}b^{-2}c^{-5}$ and $(a^{1/3})^{-5/4}$ are illustrations of kinds in which he will suffer by not exercising judgment as to method. The chapters on Theory of Exponents and on Radicals give especially favorable ground for training a pupil to find and maintain a mental equilibrium. In this connection it may be noted that frequent drill on the fact that while $(ab)^m = a^mb^m$, on the other hand $(a+b)^m$ does not equal a^m+b^m would prevent a large proportion of errors made in radical expressions. The average pupil, in these and in other instances, does not use his judgment however unless he has been trained to do so. Left to his own devices, his procedures are most haphazard. In other subjects, his judgment is exercised, to be sure, but in algebra he can experience, definitely and directly, the effect of failure to use it.

From algebra may be developed practice in generalization. In factoring, for instance, the pupil who clearly sees that an expression such as $(a-b)^2-(c-d)^2$ is essentially the same as his type form x^2-y^2 and who realizes that his factoring is accomplished when he has expressed $(a-b)^2-(c-d)^2$ as a binomial times a binomial, and that whatever else he does is in the way of simplifying those factors, and who has, furthermore, learned to group certain three, four and six term polynomials so that they present the same type, that pupil carries in his mind one kind of factoring instead of many, and he has had his perceptions quickened by his having here discerned the type among individuals. It is true that he can be trained to give with accuracy the simplified forms of the factors without so much ado. That is, given $(a-b)^2-(c-d)^2$, he may be trained to state directly the four term factors $(a-b-c+d)$ $(a-b+c-d)$ but the chances are that he gets them by some such mental path as this: "They always come out of the first parenthesis with the same signs and out of the second parenthesis, once with the same and once with the opposite signs." Of course, if the concrete result, the answer so called, is the only thing of importance, no criticism of that method is in order, provided, of course, that he is sure to remember it and provided, also, that there is no objection to lumbering his mind with two sticks when one stick would suffice. The point I am trying to make here is that whatever a pupil may do mentally, he should be able to supply the successive stages by which it is properly developed, if challenged.

Again, algebra may give development in that flexibility of mind which comes in some degree from practice in expressing the same thought in different media. Problems give an opportunity for this. The majority of pupils are studying a foreign language and they can readily be brought to regard as a vocabulary of words and phrases what is usually called the statement of the problem and to consider the equation itself as a translation from English into algebraic language. The equation can then be regarded as the expression of a thought, and not as an arrangement of x's and numbers, an arrangement chanced upon frequently after the rejection of others that refused to work out into satisfactory results, or imitated slavishly from a model, an arrangement of uncertain meaning but acceptable because from it can be obtained some desired numbers. Pupils not taking a foreign language can without much difficulty understand and adopt such a treatment of problems. The habit of before solution translating the equation back into literal English and of noting whether this literal rendering adequately expresses the problem, checks errors and tends towards further flexibility in expression.

Algebra may be so taught as to stimulate purposive thinking. Given $s = [n(a+l)]/2$, to solve for l, pupils can be trained to see that to release l from the undesired relations that hold it, they must do as they would do in the world of affairs. If they would unlock what has been locked, they must reverse the operation of locking, if they would unscrew what has been screwed, they must reverse the operation of screwing, so, to eliminate 2 from the fraction $[n(a+l)]/2$ they must determine that 2 is held there by division and must remove it by the inverse of division, multiplying it away and then, to preserve the equation, multiplying each other term, also by two. Similar purposive thinking, sufficiently continued, gives him l in the terms of the other letters. Such mental procedure is not difficult for even the average freshman mind. Training for it can begin when first he meets the equation. Accustomed to such purposive attacks as this, should the algebra of his physics so confound him as it is sometimes reported to do?

Again, it being human to wish to use and to extend a power that is felt, algebra, in so continuously setting before a pupil units of achievement, creates in him a pleasure, more or less consciously felt, in the possession of the power of achieving and fosters persistency and patience through his exercise of this power.

If the foregoing are some of our aims, shall we adapt our teaching so as to realize more of them in the average freshman? What is the average freshman? If he is the type numerically conspicuous in our classes, his most striking characteristic is his tendency to be impressed by what appears rather than by what is. He is a being with a facility (sometimes fatal to him when we do not find him out) for seeming to know what he does not know. Enchain his attention by an explanation or give him a model to follow and he will show skill in those exercises that are superficially as well as actually similar. Then direct him to some that involve the same principles but look different, such as literal forms, and he is quite likely to say he "cannot do that kind," or to do them, to his own undoing. Listen to him, the average Freshman in his natural state, as he explains to a classmate the procedure in such an equation as $x + b = c$. "You take b," he says, "and change its sign and put it on the other side"; or hear him tell another what to do in the equation $x/3 + y/2 = 5$. "You take this denominator," he says, "and multiply that numerator and take that denominator and multiply this numerator and multiply the whole number by both of 'em." It is true that we ourselves manipulate mechanically thus and find it restful to do so but along with the apparent process our minds have long since connoted the real one; hence we are guarded from deducing, as they do, incorrect procedures from mechanical practices.

The average freshman is impatient of detail in writing out his work and is prone to attempt to do mentally more than his mastery of technique warrants. While solving equations, for instance, he makes more transformations in one writing than he is capable of making accurately. Frequently, the time lost in this way, first, that used in calculating the results and, secondly, that in tracing errors when the completed work reveals that calculations were not correct, about equals the time it would take to write rapidly with little mental strain the successive transformations, one by one. He has gained nothing in time and has lost, probably, in courage. "Let your fingers be the servants of your mind" is a precept he may profitably heed throughout his freshman year. Rapidity and accuracy in transformation become possible when he proceeds one step at a time and the established habit furnishes him with drill on elements.

He has no sense of form in written work but is slipshod and fragmentary. For instance, he links equations together by equality signs and he is likely to write down the particular part of an expression that is engaging his attention and link it by an equality sign, at one or at both ends, to the whole expression—that is, he may write $16a^4 + 16a^3 + 4a^2 = 4a^2(4a^2 + 4a + 1) = (2a + 1)^2$ and $(a - b)/2 + (a + b)/3 = 3(a - b) + 2(a + b) = (5a - b)/6$.

He does not correllate. Algebra is to him a diverse multiplicity. He thinks of a certain section of the book, for instance, as "P'rentheses." He has to be told that here he is performing the same operations that he has been performing, the only distinctions being that he is translating his directions from a notation instead of seeing them in English, and is writing his results in horizontal lines. He does not reflect that an argument against thinking that $a+b$ equals $\sqrt{a^2+b^2}$ lies in the fact that $a+b$ squares into a trinomial. He does not notice that, given a new type of an equation, his method is to obtain from it one of the types already familiar; that from a linear equation of one unknown involving parentheses, he seeks, generally, one of the type $5x-2=2x+7$; that given a pair of equations in two unknowns, his concern should be to get from them one in one unknown; that from a set of three in three unknowns, he is to obtain, generally, two in two unknowns; that from a fractional equation he is to get an integral equation, and, from a quadratic, a linear.

To him, simplifying the radical $\sqrt{2/3}$ is an altogether distinct process from simplifying the radical $\sqrt{2/9}$. He has to be made to see that after the transformation of $\sqrt{2/3}$ to the $\sqrt{6/9}$ he merely has his previous case. In short, left to himself, he will tuck each captive idea into its own separate brain cell.

Finally, and in this we find our encouragement and his hope, the average freshman is a being susceptible to a considerable degree to training in clear and unified thinking. In the attempts to so train him, we may lay ourselves liable to the charge of making his work too easy for him. Our reply to such criticism may spring from the conviction that whatever we accomplish in the way of unifying and clarifying his ideas and in obtaining precision in his exercises, we cannot have enervated him if we have given him the freedom and the desire to use more mental powers than he otherwise would use.

What of our presentation to him of the subject matter? Are our sequences advantageous to him? We are under no necessity to keep the pace and the route of our particular text-book since many of the exercises in algebra can be taken out of the order in which they are set. Wherever it is possible by a change in arrangement to gain in time or in economy of the pupils' effort, that change is worth making. You need leisure at certain stages in teaching algebra and you can secure it in this way.

For instance, some books introduce multiplication by the case of a monomial times a monomial, following this by monomial times polynomial, before giving polynomial times polynomial, making two lessons prior to the last case. Those two lessons are not needed. A few minutes' oral drill on monomial times monomial, a few moments' attention to the likenesses and dissimilarities between the methods of operation in arithmetic and in algebra and the class can be launched successfully upon multiplying polynomial by polynomial. The case of adding a whole number and a fraction needs no individual treatment but can be brought under the addition of fractions by regarding the whole number as the numerator of a fraction whose denominator is one, the addition thus becoming that of fractions. Literal equations need not be considered by themselves. In taking up new types of equations, we can assign in each lesson a few literal ones along with the numerical. Addition need not be given the complicated consideration it receives in some books. One of the recent admirable text-books sets it forth under three captions: To add two or more positive numbers; To add two or more negative numbers; To add a positive and a negative number. Now a pupil can easily manage all these without difficulty—as one case, by being told to add positive and negative numerals as he would find a man's financial condition from his liabilities and his assets. If he questions, "When you add in algebra, do you sometimes subtract and sometimes add," he is satisfied by the reply, "When you add numerals in algebra, you combine them as you would a man's debts and resources." The phrase "algebraic sum" is found in many text-books. Is it not conducive to a pupil's thinking that adding in arithmetic is essentially different from adding numerals in algebra?

Multiplication can to advantage be taught directly after addition. Before the mind is confused by subtraction, the inverse of addition, and with very little expenditure of time or effort in teaching multiplication, further drill in addition can be found in summing the partial products. The interpretation of notations can join with further drill on multiplication by the introduction at this point to parenthetical expressions involving only multiplication. If any one questions, as I have several times been asked, how explain that minus times minus equals plus without an understanding of subtraction, he may be referred to J. W. Young's "Fundamental Concepts of Algebra and Geometry" and to Fine's "The Number System of Algebra." A five minute game, minus times plus, plus times minus, minus times minus, times plus, times plus, times—minus, minus times—minus, minus times minus, minus times plus, etc., fixes the law satisfactorily for them. Also, multiplication and division of fractions can follow reduction of fractions to lowest terms, since they involve so little more teaching and furnish so abundant practice in reduction. By the way, why do we allow pupils to follows book models in the multiplication of fractions, rewriting each fraction with its numerator and denominator in prime factor form and then cancelling criss cross from one numerator to another denominator, multiplying finally the remnants of the fractions. It is an unsightly method and it does not accord with any rules or principles they have had. They are not formally multiplying the given fractions and they are not formally reducing any single fraction to its lowest terms. That principles might be furnished them which would cover such a procedure seems beside the question. If we are training them for any thing but concrete results why not hold them to a regard for what is within their knowledge. By multiplication of the fractions directly into one fraction, expressing numerator and denominator of the product in their factored forms, and making a reduction of that fraction to lowest terms they act within their rules and use a more compact form. Purposive thinking in this connection may raise to them this question, "Why multiply numerator and denominator under a factorial rather than under a simplified form?"

After long division has furnished a drill on subtraction, any lingering confusion between the processes of addition and subtraction can be clarified in simultaneous linear equations, in two and in three unknowns, the class thus gaining considerable technique before going further. The graph and the indeterminate equation introduced at this point, in checking one another and in giving drill on the substitution of negative numbers, furnish them with easily determined tests of their own accuracy. Besides, if pairs of equations are now given to be solved simultaneously, indeterminately, graphically, the pupil gets a drill on passing rapidly through a succession of different operations when each operation is one in which he is fairly accurate. Without this, unprepared for much diversity, he enters shortly upon addition of fractions where the rapid passing through a variety of processes, some not yet sufficiently drilled, to wit; recognizing factors, determining the least common multiple of denominators, reduction to equivalent fractions having their least common denominator, addition into a single fraction, simplification of the numerator, and reduction to lowest terms, all distract his attention from accuracy.

Such changes of order as these result in furnishing some of the material for drill which text-books lack. Mathematical teaching as a rule compares unfavorably with that of music and languages in the matter of drill on elements.

We can clarify some ideas for the average pupil by refusing to permit in the class room words that connote what is mechanical rather than mathematical. Cancel and transpose are two such terms. The average pupil delights in cancelling. There is something in the bold sweep of the arm as he dashes out this, out that, something in this summary rejection of what he would eliminate, that endears it to him. But what does he mean when he says he cancels? When he has $x^2+6x=x^2+8$ he says x^2 cancels. Here he should mean that he adds—x^2 to each member of his equation. He says when he has $[x^2(a+b)]/[x^2(a-b)]$ that x^2 cancels Here he should mean that he divides numerator and denominator both by x^2. As a matter of fact all he does mean is that in each case x^2 occurs in two places and that he is pleased to rid himself of it by crossing it out. Small wonder that soon he crossed out x in $(b+x)/ax$. If he is not quite so reckless as this, he is a very unusual pupil if at some time he does not make the same sort of an error in a more complicated expression. Such mistakes are common among advanced pupils of some attainment. We can minimize their frequency by explicitly instead of tacitly teaching that the removal of a factor from a product is a method of division by that factor, by the discussion of possible transformations of a fraction and by seeing to it, through frequent challenge and cross question, that, when they reduce fractions to lowest terms, they are intelligently following the rule which they are taught; namely, "divide numerator and denominator by their highest common factor." Transpose is another misleading term, accountable for thoughtless work. For them it means, change the sign and move. Let me illustrate from the work of a boy who first appeared to me in the middle of his junior review. He is faithful, attentive, fairly accurate and intelligent. In one of his previous tests I had noticed badly mangled equational forms and had criticized them. This, therefore, seems to have resulted from no momentary aberration, such as even the best of pupils may experience. The equation was $x - 3y = 5$, and his expression for y, $y = 8 - x$. His mind must have worked thus: $x - 3y = 5$, x is in the way on the left side so change its sign and put it on the right, getting $-3y = 5 - x$. -3 is also in the way on the life side, so also change its sign and put it on the right. $-3y = 5 - x$, $y = 5 - x + 3$ and $y = 8 - x$.

We can clarify ideas by letting the mind derive them from notations rather than from a variety of rules. Is not addition in algebra a matter of notation and is not much that goes under the name of addition, the simplifying of a sum by means chiefly of monomial factoring, through, of course, the distributive, associative and commutative laws. We add $6a$ and $-2b$ by writing $6a - 2b$. We can add $6a$ and $-2a$ by writing $6a - 2a$. But because the latter can be rewritten $(6 - 2)a$ and then as $4a$, we get a simpler form. It may not seem wise at the beginning of algebra to teach additions of polynomials thus subtly, but when addition of fractions is reached, a brief explanation of what underlies the addition of polynomials can introduce the class to such an understanding of the method for the addition of fractions. If he is first drilled a little on the fact that a fraction may factor into its numerator and the reciprocal of its denominator, he can see why he reduces his fractions to equivalent fractions having their least common denominator, he can then factor his polynomial into the reciprocal of the least common denominator and the resulting quotient. This resulting quotient, if he has been well drilled on simplifying expressions containing parentheses involving both subtractions and multiplications, he can readily set down as a factor free from compound terms; then, if he has already had multiplication of fractions and if he regards the quotient factor above referred to, since it is an integer, as the numerator of a fraction whose denominator is unity, the rest of the simplification becomes the multiplication of two fractions. My experience with this method of adding fractions has been most satisfactory. Pupils take hold of it readily from the first. When they take up transformation of fractional to integral equations, they no longer are in danger of confusing the two, and consequently of dropping off denominators when they add fractions and their work on complex fractions by the methods involving addition is very straightforward and clear. Of course in complicated expressions, such a method, because of the parentheses it involves, might seem to be something of a nuisance but by the time pupils are able to deal with such complicated expressions they can perform mentally that step in which this method differs from the conventional one, namely the step where the factoring is done, leaving the form of the work the conventional form. And the addition of radicals, by this same method of factoring, becomes the simplification of radical terms of a polynomial sum, and the substitution of a factorial form for every group having a common surd factor. Such a method does not involve, as does that in many books, taking the polynomial to pieces and later putting it together again, frequently with some injury of signs. Much of the difficulty of highest common factor and least common multiple can be removed by insistence on interpretation of these words themselves. No rule is necessary for the highest common factor if the pupil can be brought to see that he needs only to determine what are the factors that are common to all the expressions. The word common, by the way, usually is without significance to him. Its meaning in this connection can be lodged in his mind by discussing its application in the term, Boston common. The least common multiple can be handled without intervention of a rule if the pupils can be made to see, by arithmetical and algebraic illustrations, first, what a multiple is, and secondly, that a multiple of an expression must be a multiple of every factor of that expression.

All he needs to know about exponents, before he comes to the chapter on their theory, is contained in the statement $a^2a^3 = aa aaa = a^5$. When he asks, "Do you add the exponents when you multiply a^3 and a^4," a sufficient reply is, "You know $a^2a^3 = aa\,aaa = a^5$, what do you think a^3a^4 can mean?" As he has, in other ways, much practice in addition, why distract his mind by mechanical rules, almost necessary if we introduce him to literal integral exponents before we come to the chapter on the theory of exponents. When he reaches this chapter, eight statements furnish him with a sufficient working basis for his needs. These are,

as $a^2a^3 = aa\,aaa = a^5$, so $a^ma^n = a^{m+n}$
as $(a^2)^3 = aa\,aa\,aa = a^6$, so $(a^m)^n = a^{mn}$
as $(a/b)^2 = a/b \times a/b = a^2/b^2$, so $(a/b)^m = a^m/b^m$
as $(ab)^2 = ab\,ab = a^2b^2$, so $(ab)^m = a^mb^m$.
If $a^{2/3}a^{2/3}a^{2/3} = a^2$, then $a^{2/3} = \sqrt[3]{a^2}$, and similarly, $a^{m/n} = \sqrt[n]{a^m}$.
If also $a^{m/n} = (a^{1/n})^m$, then $a^{m/n} = (\sqrt[n]{a})^m$.
If $a^0a^m = a^m$, then $a^0 = 1$.
If $a^{-m}a^m = 1$, then $a^{-m} = 1/a^m$ and $a^m = 1/a^{-m}$.

His accuracy can be increased by providing him with checks upon his possible errors. There are many checks besides that by numerical substitution. It is a question to what extent the weaknesses of a pupil in arithmetic should discourage his progress in algebra. When a numerical check fails to verify his answer, the error frequently is not in the algebraic work. If the desire is to use the freshman algebra to gain power in numerical computation, much practice of this sort is worth while. But for gaining technique in algebra there are other checks more valuable, in that they locate the error and in that the check itself is not so likely to be erroneous. For instance, errors in subtractions may be checked by mental additions of the remainder to the subtrahend. Classes trained to this habit will not present long division examples perforated with many errors in subtraction and yet, in their parlance, "coming out even." The correctness of signs in an expression put into a negative parenthesis may be tested by a mental removal of the expression from the parenthesis. Multiplication after each factoring checks errors there. This is of particular advantage when the factoring is first taught. If the practice is insisted on, the pupil will at once have means of determining what types he has or has not understood and where he is or is not careless in using them. When fractions have been reduced to equivalent fractions having their least common denominator, the mental reduction of each fraction to its lowest terms and the comparison of the result with the original fractions show up mistakes when they occur. In solving problems, the habit of translating the equation back into English to see if a literal rendering is the equivalent of what was given in the problem, guards against the frittering away of time in solving useless equations. In finding highest common factors and least common multiples, the challenging of the results by certain queries can bring common sense into the lists. Some such queries are: Is each of these factors common to all the given expressions? Does this, which I am calling the highest common factor, contain all the factors common to all the given expression? Is this a multiple of each factor of each of the given expressions? Does it contain an unnecessary factor?

Algebra differs from most other subjects, excepting the languages, in that it is cumulative, so that if a pupil once gets below the class average he is at a greater disadvantage because of this. There is therefore this added reason for making it worth while to consider as many ways as possible for keeping a pupil in the game. When he has no confidence in his own accuracy, no means of testing it, when he has no way of judging whether to attribute failure to lack of understanding or to lack of care, what wonder that he becomes discouraged when he is entangled in his mistakes. Unsuccessful mental effort is far more fatiguing than successful mental effort. The more possible that accurate manipulation is made to the average freshman, by drill on elements and by training him to gauge his own results, the more courage and interest he will have for his daily encounter with his lesson. Besides, we are learning to use prophylactic measures against disease and crime. Why not use them in pedagogy?

My statements regarding the natural tendencies of the average freshman have been derived during the last fifteen years from interested and analytical reading of the test papers of some hundreds over a thousand freshman pupils and of many boys and girls whom I have first met in review algebra. Each departure that I have advocated from the conventional book method I have worked out in the class room at least several times and have found practicable for freshmen. My deep convictions are that, in the same course both classes of our pupils may contentedly satisfy their individual needs, the one for the content of algebra, the other for the training experienced through it, and that the average freshman may participate to a considerable degree in clear and unified thinking.

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